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Reconstructing a Function from Its Values on a Subset of Its Domain—A Hilbert Space Approach

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1. INTRODUCTION

An important problem area, encompassing many special questions which have been treated in the literature, is that of characterizing the restrictions of functions in some prescribed class to a subset of their domain, and where possible, "reconstructing," i.e., extrapolating a function from its restriction. For example, if $\mathbb{D}$ denotes the unit disc in the complex plane and $E \subset \mathbb{D}$ we can ask, given a complex-valued function φ on E, whether or not there is a function f in the Hardy space $H^2(\mathbb{D})$ satisfying

$$f(z) = \varphi(z), \qquad \text{all} \quad z \in E. \tag{1.1}$$

This question has two aspects, which of course are intimately connected:

(i) determining whether such an f exists and

(ii) finding a method to calculate f when it exists (of course, for f to be *unique* will require that E be, in some sense, sufficiently large).

Typically, problems of this kind are "ill posed" in the sense that the set of functions φ on E for which an f exists satisfying (1.1) is "unstable with respect to small perturbations," i.e., fails to be an open set in most reasonable topologies, and this motivates problems of the type

(iii) Let μ be a positive measure on E, and φ a given complex-valued function in $L^2(E; d\mu)$. For a given positive number M, find that $f \in H^2(\mathbb{D})$ with $\|f\| \leqslant M$ such that

$$\int_E |f - \varphi|^2 \, d\mu$$

is minimum.

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(Of course, we need assumptions implying that the restriction map $f \mapsto f|_E$ is continuous from $H^2(\mathbb{D}) \to L^2(E; d\mu)$ if this problem is to make sense. Since the above model problem is here used only to illustrate our general purpose we will not now examine such technical points more closely.)

In Section 2 of this paper we study an abstract problem in which the role of the above restriction map is played by a bounded linear operator T from a Hilbert space X to a Hilbert space Y, which moreover always will be assumed *injective, with dense range*. That these hypotheses are satisfied in many "restriction-extension problems" of the above type that have been studied in the literature will be seen in Section 3, where various concrete problems are examined in the light of the general results obtained in Section 2. Section 3 should be regarded mainly as programmatic, to illustrate the kinds of special problems that fit into our general framework and also to pinpoint concrete problems that seem of interest for detailed investigation later.

In the case where T is compact the analogous theory was developed (with different notations) in [12]. There the spectral resolution of T^*T in terms of its eigenfunctions

$$T^*Tx_n = \lambda_n x_n \tag{1.2}$$

and correspondingly

$$TT^*y_n = \lambda_n y_n, \qquad y_n = Tx_n \tag{1.3}$$

played the central role; in concrete problems where T is a restriction operator the $\{x_n\}$ correspond to an orthonormal basis in the basic space X (which would be $H^2(\mathbb{D})$ in the above example) and then the orthogonality of the $y_n = Tx_n$ means that these functions (or rather, their restrictions) are also orthogonal in a second sense (in the space $L^2(E; d\mu)$, in the above example). Eqs. (1.2) and (1.3) which, on the one hand, explain this phenomenon of "double orthogonality" that crops up in so many places, and also imply a constructive proof of the unitary equivalence of T^*T and TT^*, are in this case at least *formally* simple due to the compactness of these operators (although *finding* the eigenvalues and eigenfunctions in concrete problems is usually difficult, and seldom possible in terms of known special functions).

However, when T^*T has continuous spectrum even the formal aspects of the problem are much less simple. It was remarked in [12, p. 53] that the solution of problems of the type (i) and (ii) above could in principle be given in terms of the resolution of the identity on X induced by the self-adjoint operator T^*T (or, that on Y induced by TT^*). Below we present the details of this solution as well as that of the abstract problem of type (iii) above, i.e., for given $y \in Y$:

(iii)′ *minimize* $\|Tx - y\|$ *over* $\{x \in X: \|x\| \leqslant M\}$.

After the completion of the research embodied in this paper, I found that problem (iii)′ had been solved earlier by Rosenblum [11]. I take the liberty of including my proof of one of Rosenblum's results here because it yields the result in a slightly different (of course, equivalent) from that is most convenient for my purposes. Incidentally, Rosenblum's results show, in a sense, why the adjoint operator T^* plays such an important role in problems of type (i) and (ii) above (which otherwise might seem rather mysterious): T^* arises naturally in the solution of problem (iii′), and problems of type (i) and (ii) can then be interpreted as studying the limiting behaviour of the minimizing element $x = x_M$ as $M \to \infty$. Theorem 1 could easily be deduced from Theorem 2 (due to Rosenblum) but we have preferred to give a direct proof independent of variational problems.

Our exposition is based on the spectral theorem for bounded self-adjoint operators on a Hilbert space and the "functional calculus" expressing bounded Borel functions of the operator in integral form (thorough treatments of this may be found, e.g., in the books of Riesz–Nagy [10] or Stone [1]). Presumably most of our results could be pushed through for unbounded operators, however, that will be left for a future investigation. On the other hand, all our results are severely limited to the Hilbert space framework, so that "reconstruction" problems for H^p, bounded analytic functions etc. as studied, e.g., in [4, 5, 8, 12] fall outside the scope of the present paper. Finally, let me emphasize that the aim of this paper is unification. It shows the "common denominator" in such apparently diverse investigations as Bergman's doubly orthogonal functions [2], the Slepian–Pollak theory of band-limited functions [14], Krein and Nudelman's reconstruction of an H^2 function in a half-plane from its boundary values on an interval [6] as well as studies to the same end by Steiner [15, 16] and van Winter [18], Patil's Toeplitz operator method [10], and Mats Lindberg's (unpublished) studies of reconstructing an H^2 function in the unit disc from its values along a diameter and similar problems. For example, our study reveals that the Slepian–Pollak integral operator gotten by first time-limiting and then band-limiting (see Sect. 3) plays exactly the same role as Patil's Toeplitz operator whose symbol is a characteristic function of a set -a fact which is easy to miss in view of the very different contexts of those papers. Thus, while the abstract theory in Section 2 qua Hilbert space theory is rather simple and perhaps overlaps in places investigations carried out by others with different purposes in mind I feel that its unifying role with regard to the circle of problems enumerated above gives it some measure of novelty and interest.

2. The Abstract Framework

Throughout this section X, Y denote complex Hilbert spaces and $T: X \to Y$ *shall always denote a bounded linear operator which satisfies*

$$T \quad \textit{is injective} \tag{2.1}$$

and

$$\operatorname{im} T \quad \textit{is dense in } Y, \tag{2.2}$$

where, as usual im T ($= TX$) denotes the range of T. Then T^* is a bounded linear operator from Y to X, characterized by

$$\langle Tx, y \rangle_Y = \langle x, T^*y \rangle_X,$$

where $\langle\ ,\ \rangle_X$ and $\langle\ ,\ \rangle_Y$ denote the inner products in X, Y respectively, and from (2.1), (2.2) follow in turn that T^* *has dense range in* X, *and is injective.* Likewise the bounded self-adjoint operators A on X, and B on Y defined by

$$A := T^*T \tag{2.3}$$

$$B := TT^* \tag{2.4}$$

are injective (indeed, strictly positive) and have dense range. Letting $E(\cdot)$ and $F(\cdot)$ denote the resolutions of the identity on X, Y, respectively, corresponding to A and B we have then, by "functional calculus"

$$f(A) = \int_0^{L+0} f(\lambda)\, dE(\lambda) \tag{2.5}$$

for all f continuous on $[0, L]$ where $L = \|A\|$ (and more generally for bounded Borel functions) and a similar formula involving B.

Note. in what follows we will usually omit the subscripts in notations like $\langle\ ,\ \rangle_X$ or $\|\cdot\|_Y$ since the context will make the notations unambiguous. In like manner we shall denote indifferently by I the identity operators both in X and Y. Also, sp A denoes the spectrum of a linear operator A.

Theorem 1. *With* T *as above, for every* $y \in Y$ *the following are equivalent*:

(i) $y \in \operatorname{im} T$

(ii) *Defining* x_c *for* $c > 0$ *by*

$$x_c = T^*(TT^* + cI)^{-1}y, \tag{2.6}$$

$\|x_c\|$ *are bounded for* $c > 0$.

(iii)

$$\int_0^{M+0} \lambda^{-1}\, d\|F(\lambda)\, y\|^2 < \infty, \tag{2.7}$$

where $M = \|TT^*\|$.

(iv) *There is an* $x \in X$ *such that*

$$\lim_{c \searrow 0} \|x_c - x\| = 0. \tag{2.8}$$

In this case, moreover,

$$\|x\|^2 = \int \frac{d\|F(\lambda)\, y\|^2}{\lambda} \tag{2.9}$$

and $Tx = y$.

Proof. (i) $\Rightarrow$ (ii) Assume $y = Tx$, $x \in X$. Then

$$x_c = T^*(TT^* + cI)^{-1}\, Tx = T^*T(T^*T + cI)^{-1}\, x. \tag{2.10}$$

(Here we have used the identity

$$(TT^* + cI)^{-1}\, T = T(T^*T + cI)^{-1}$$

which follows from the obvious identity

$$T(T^*T + cI) = (TT^* + cI)\, T$$

upon multiplying both sides by $(TT^* + cI)^{-1}$ on the left, and $(T^*T + cI)^{-1}$ on the right.) From (2.10) we have $x_c = f(A)\, x$, where $f(\lambda) = \lambda(\lambda + c)^{-1}$. By "functional calculus," $\|f(A)\|$ cannot exceed the maximum of $f(\lambda)$ on sp A, hence $\|f(A)\| \leqslant 1$ and so $\|x_c\| \leqslant \|x\|$.

(ii) $\Rightarrow$ (iii) Assume $\|x_c\| \leqslant K < \infty$. Now,

$$\begin{aligned} \|x_c\|^2 &= \langle T^*(TT^* + cI)^{-1}y,\ T^*(TT^* + cI)^{-1}y \rangle \\ &= \langle TT^*(TT^* + cI)^{-1}y,\ (TT^* + cI)^{-1}y \rangle \\ &= \|z\|^2, \end{aligned}$$

where

$$z = g(B)\, y,$$
$$g(\lambda) = \lambda^{1/2}(\lambda + c)^{-1}.$$

Hence

$$\|x_c\|^2 = \int \lambda(\lambda + c)^{-2} \, d\|F(\lambda)\, y\|^2 \leqslant K \tag{2.11}$$

and, letting $c \searrow 0$, Fatou's lemma gives (2.7).

(iii) $\Rightarrow$ (iv) Assume now that (2.7) holds. From the first equality in (2.11) we see that $\{x_c\}$ is bounded. Hence there is a sequence $\{c_i\}$ with $c_i \searrow 0$ such that $\{x_{c_i}\}$ converges weakly, say

$$x := \text{weak lim } x_{c_i}.$$

Hence

$$\begin{aligned} Tx &= \text{weak lim } TT^*(TT^* + c_i I)^{-1}\, y \\ &= B(B + c_i I)^{-1} y + w_i, \end{aligned}$$

where $w_i \to 0$ weakly in Y. Hence

$$(B + c_i I)\, Tx = By + Bw_i + c_i w_i$$

and now $i \to \infty$ gives $BTx = By$, whence $y = Tx$. Therefore (2.10) holds, and

$$\begin{aligned} x - x_c &= c(T^*T + cI)^{-1}\, x, \\ \|x - x_c\|^2 &= \int \left(\frac{c}{c + \lambda}\right)^2 d\|E(\lambda)\, x\|^2, \end{aligned}$$

and by Lebesgue's bounded convergence theorem the last integral tends, as $c \searrow 0$, to zero. (Note that for this reasoning to be valid it is necessary that the measure $d\|E(\lambda)\, x\|^2$ place no mass at $\lambda = 0$, and thus is a consequence of the injectivity of T^*T.) Finally, (2.9) is an evident consequence of (2.11), (2.7) and Lebesgue's bounded convergence theorem. This completes the proof of Theorem 1.

Remark. The introduction of the elements

$$x_c = T^*(TT^* + cI)\, y = (T^*T + cI)^{-1}\, T^*y \tag{2.12}$$

is naturally motivated by the observation that (under our assumptions) $Tx = y$ is equivalent to $T^*Tx = T^*y$. This suggests *formally* $x = (T^*T)^{-1}$ T^*y; while this is not meaningful when T fails to be invertible it does strongly suggest (2.12). Actually, more is true, as Rosenblum discovered: x_c has an *extremal property*, as indicated in the next theorem.

THEOREM 2 (Essentially, Rosenblum [10]). *For any $y \in Y$ with $y \neq 0$ let*

$$M_0 = M_0(y) := \left(\int \lambda^{-1} \, d\|F(\lambda)\, y\|^2 \right)^{1/2}$$

(so that $0 < M_0(y) \leqslant \infty$). For any M, $0 < M < M_0$ there is a unique element x_M lying in

$$B_M := \{x \in X \colon \|x\| \leqslant M\}$$

minimizing $\|Tx - y\|$. Moreover,

$$x_M = T^*(TT^* + cI)^{-1}\, y = (T^*T + cI)^{-1}\, T^*y,$$

where $c = c(M)$ is uniquely determined from the equation

$$\int \lambda(\lambda + c)^{-2} \, d\|F(\lambda)\, y\|^2 = M^2 \tag{2.13}$$

Remark. Clearly the left side of (2.13) increases strictly from 0 to M_0^2 as c decreases from $+\infty$ to 0. Note also that if $M_0 < \infty$ then, in view of Theorem 1, the minimum of $\|Tx - y\|$ for $x \in B_{M_0}$ is zero (and of course, it is zero for $x \in B_M$, $M > M_0$). Theorem 2 can be viewed as a quantitative sharpening of Theorem 1 whose essence was:

$$y \in \operatorname{im} T \qquad \text{if and only if } M_0(y) < \infty.$$

Proof of Theorem 2. Since the function

$$x \mapsto \|Tx - y\|^2$$

is strictly convex on X, and moreover lower semicontinuous on B_M when this is given the weak topology induced by X (so that B_M is then compact) the existence of a unique minimizing element x_M follows. We show first

$$\|x_M\| = M. \tag{2.14}$$

Indeed, were this not so then for small complex λ and arbitrary $z \in X$, $x_M + \lambda z$ would be in B_M and so

$$\|T(x_M + \lambda z) - y\|^2 \geqslant \|Tx_M - y\|^2$$

which implies $\langle Tx_M - y, Tz \rangle = 0$ for all z, i.e., $Tx_M = y$. We know from Theorem 1, however, that $Tx = y$ is not solvable for $x \in B_M$ with $M < M_0$. Thus (2.14) holds, and *in particular*, x_M *is not* 0.

Let now $z \in X$, $\lambda \in \mathbb{C}$ and

$$w := M\|x_M + \lambda z\|^{-1}\, (x_M + \lambda z)$$

so that w is well defined and in B_M if $|\lambda|$ is small enough. A simple calculation shows that

$$w = x_M - M^{-2}\operatorname{Re}\langle x_M, \lambda z\rangle\, x_M + \lambda z + O(|\lambda|^2)$$

for small $|\lambda|$. Let us choose z so that $\langle x_M, z\rangle = 0$, so

$$w = x_M + \lambda z + O(|\lambda|^2)$$

and so

$$\begin{aligned}\|Tx_M - y\|^2 &\leqslant \|Tw - y\|^2\\ &= \|Tx_M - y\|^2 + 2\operatorname{Re}\langle Tx_M - y, \lambda Tz\rangle + N|\lambda|^2\end{aligned}$$

for some constant N, whence

$$0 \leqslant 2\operatorname{Re}(\bar{\lambda}\langle Tx_M - y, Tz\rangle) + N|\lambda|^2.$$

By varying the argument of λ we get

$$0 \leqslant -2|\lambda| \cdot |\langle Tx_M - y, Tz\rangle| + N|\lambda|^2$$

whose validity for all small $|\lambda|$ implies $\langle Tx_M - y, Tz\rangle = 0$. Hence $T^*Tx_M - T^*y$ is orthogonal to every vector z that satisfies $\langle x_M, z\rangle = 0$ and so

$$T^*Tx_M - T^*y = -cx_M \tag{2.14}$$

for some complex constant c.

We shall show momentarily that $c > 0$. Assuming this for the present we get

$$x_M = (T^*T + cI)^{-1}T^*y = T^*(TT^* + cI)^{-1}y$$

and now, since

$$\begin{aligned}M^2 = \langle x_M, x_M\rangle &= \langle TT^*(TT^* + cI)^{-1}y, TT^* + cI)^{-1}y\rangle\\ &= \int \lambda(\lambda + c)^{-2}\, d\|F(\lambda)\, y\|^2,\end{aligned}$$

(2.13) is verified, so the proof will be complete once we show $c > 0$.

Verification that $c > 0$. Taking inner products of both sides with x_M in (2.14) gives

$$\langle Tx_M - y, Tx_M\rangle = -cM^2. \tag{2.15}$$

Now, $c=0$ is impossible since then (2.14) would yield $T^*(Tx_M-y)=0$ and, in view of the injectivity of T^*, $Tx_M=y$ which we know is impossible for $M<M_0(y)$. So, we need only show $c\geqslant 0$ or, in view of (2.15), that

$$\|Tx_M\|^2\leqslant\langle y, Tx_M\rangle \tag{2.16}$$

holds. Now, for complex α, $|\alpha|\leqslant 1$, $\alpha x_M\in B_M$ and so

$$\|T\alpha x_M-y\|^2\geqslant\|Tx_M-y\|^2,$$
$$|\alpha|^2\|Tx_M\|^2-2\operatorname{Re}(\alpha\langle Tx_M, y\rangle)+\|y\|^2$$
$$\geqslant Tx_M\|^2-2\operatorname{Re}\langle Tx_M, y\rangle+\|y\|^2,$$

whence

$$(1-|\alpha|^2)(\|Tx_M\|^2+2\operatorname{Re}(\alpha\langle Tx_M, y\rangle)\leqslant 2\operatorname{Re}\langle Tx_M, y\rangle. \tag{2.17}$$

First, let $|\alpha|=1$. Then by varying $\arg\alpha$ we get

$$|\langle Tx_M, y\rangle|\leqslant\operatorname{Re}\langle Tx_M, y\rangle$$

so we conclude

$$\langle Tx_M, y\rangle \qquad \textit{is real and } \geqslant 0.$$

Now choose $0<\alpha<1$ in (2.17). We get

$$(1-\alpha^2)\|Tx_M\|^2\leqslant 2(1-\alpha)\langle Tx_M, y\rangle.$$

Dividing by $1-\alpha$ and letting $\alpha\nearrow 1$ now gives (2.16), and Theorem 2 is completely proved.

Remark. The variational arguments used in this proof are of course familiar in principle. Apart from Rosenblum's paper [11] similar ideas appear in earlier work of Davis [3] and many others in the solution of similar but more special quadratic minimum problems.

THEOREM 3. *There is a unitary operator U mapping X onto Y such that*

$$UT^*T=TT^*U \tag{2.18}$$

Proof. The following proof was kindly pointed out to me by Lars Svensson; it is motivated by the "polar decomposition" of a linear operator, or more precisely by the well-known fact that if T were invertible then $T(T^*T)^{-1/2}$ would be a unitary operator U satisfying (2.18), as one easily checks.

Define an operator U on the vector space $S:=\operatorname{im}(T^*T)^{1/2}$ by

$$U(T^*T)^{1/2}x=Tx. \tag{2.19}$$

Our hypotheses (2.1), (2.2) imply easily that S is dense in X. Now, for any $\xi = (T^*T)^{1/2}x$ in S,

$$\|U\xi\|^2 = \|Tx\|^2 = \langle T^*Tx, x\rangle = \|\xi\|^2.$$

Hence U is an isometric map from S to im T, and by continuity has a unique extension (still denoted by U) which maps X isometrically into Y. Since im U is closed and contains im T, it is all of Y, so U is unitary.

Finally, since

$$U(T^*T)^{1/2} = T$$

we have

$$U(T^*T) = T(T^*T)^{1/2},$$

hence

$$U(T^*T)\,U^* = T(T^*T)^{1/2}\,U^* = TT^*$$

which is equivalent to (2.18).

Remark. Another proof of Theorem 3 (my original proof) is based on a different way to imitate the nonexistent operator $T(T^*T)^{-1/2}$, namely consideration of the operators

$$U_c := T(T^*T + cI)^{-1/2}, \qquad c > 0$$

from X to Y. We obtain by calculations similar to those used in proving Theorem 1 that

$$\|U_c x - U_{c'}x\|^2 = \|z\|^2,$$

where $z = f(A)\,x$ and

$$f(\lambda) = \lambda^{1/2}[(\lambda + c)^{-1/2} - (\lambda + c')^{-1/2}].$$

Hence

$$\|U_c x - U_{c'}x\|^2 = \int \lambda[(\lambda + c)^{-1/2} - (\lambda + c')^{-1/2}]^2\, d\|E(\lambda)\,x\|^2$$

$$\leqslant (1/4)(c - c')^2 \int \lambda^{-2}\, d\|E(\lambda)\,x\|^2.$$

Now, the last integral is finite for x in a dense subset L of X, and so

$$\lim_{c \searrow 0} U_c x =: Ux$$

exists for $x \in L$. Further calculations show that U is isometric on L and has range dense in Y, and therefore extends to a unitary map from $X \to Y$ which satisfies (2.18). We leave the remaining details of this variant to the reader.

THEOREM 4. *Let K denote the spectrum of T^*T (which coincides with that of TT^* in view of the unitary equivalence of these operators asserted by Theorem* 3). *For every bounded Borel function f on an interval including K we have*

$$Tf(T^*T) = f(TT^*)\, T. \tag{2.20}$$

Proof. From the obvious identity

$$T(T^*T)^n = (TT^*)^n\, T$$

we get (2.20) for polynomials f. The general case follows from the fact that to every bounded Borel function f on a bounded interval J there is a sequence of polynomials $\{p_j\}$ such that $p_j(x) \to f(x)$ for $x \in J$ and the p_J are uniformly bounded on J, together with standard facts from "functional calculus" of self-adjoint operators.

COROLLARY 1. *For any Borel set $\Delta \subset \mathbb{R}$ the projectors $E(\Delta)$, $F(\Delta)$ corresponding to the resolutions of the identity determined by T^*T, TT^* on X, Y respectively satisfy*

$$TE(\Delta) = F(\Delta)\, T \tag{2.21}$$

Proof. Obviously we may assume Δ is bounded. Choose for f in (2.20) the function equal to 1 on Δ and 0 elsewhere. Formulas relating $E(\Delta)$ and $F(\Delta)$, somewhat different from (2.21), and suggested to me by Mats Lindberg, are given in

COROLLARY 2. *Let Δ be a Borel set of positive real numbers whose closure does not contain* 0. *Then*

$$F(\Delta) = T\left(\int_\Delta \lambda^{-1}\, dE(\lambda)\right) T, \tag{2.22}$$

$$E(\Delta) = T^* \left(\int_\Delta \lambda^{-1}\, dF(\lambda)\, T. \tag{2.23}$$

Proof. Clearly it suffices to prove (2.23). Choose, in Theorem 4,

$$f(\lambda) = \lambda^{-1} 1_\Delta(\lambda),$$

where 1_Δ is the characteristic function of Δ. We get, applying T^* on the left to both sides of (2.20)

$$T^*Tf(T^*T) = T^*f(TT^*)\,T$$

whence

$$1_\Delta(T^*T) = T^*\left(\left[\int f(\lambda)\,dF(\lambda)\right]\right]T$$

which is (2.23).

Remark. The following related fact is sometimes useful: *if R is a linear operator on Y that commutes with TT^* then T^*RT is a linear operator on X that commutes with T^*T.* The proof of this (and the analogous fact with roles of X and Y reversed) is obvious.

Finally, the following elementary result gives a version of "double orthogonality" in the case where TT^* need not be compact.

THEOREM 5. *If X_1 is a subspace of X invariant under T^*T, and x_1, x_2 are vectors in X such that $x_1 \in X_1$, $x_2 \in X_1^\perp$, then*

$$\langle Tx_1, Tx_2\rangle = 0.$$

Proof. We have

$$\langle Tx_1, Tx_2\rangle = \langle T^*Tx_1, x_2\rangle = 0,$$

since by hypothesis $T^*Tx_1 \in X_1$.

COROLLARY. *If x_1, x_2 are eigenvectors of T^*T corresponding to different eigenvalues then*

$$\langle x_1, x_2\rangle = \langle Tx_1, Tx_2\rangle = 0.$$

In concluding this section, observe that the results in [13] when TT^* is compact are special cases of those in the present paper. For example, if TT^* is compact with eigenvalues $\{\lambda_n\}$ and corresponding orthonormal eigenvectors $\{e_n\} \subset Y$, then (2.7), the necessary and sufficient condition for y to be in the range of T, becomes

$$\sum \lambda_n^{-1} |\langle y, e_n\rangle|^2 < \infty$$

in conformity with the "abstract Bergman theorem" of [13] (where the notations are somewhat different).

Note also that there exist other necessary and sufficient conditions for $y \in \operatorname{im} T$ besides that given by Theorem 1. A very simple one, valid for any bounded linear operator $T: X \to Y$, which was employed in [7], is

PROPOSITION. *A necessary and sufficient condition for y to lie in* $\operatorname{im} T$, *where T is any bounded linear transformation from $X \to Y$, is the existence of a constant M such that*

$$|\langle y, z\rangle| \leqslant M\|T^*z\| \tag{2.24}$$

for every $z \in Y$.

This criterion (whose very simple proof we omit) extends to unbounded operators, and also (with slight reformulation of (2.24)) to the case where X and Y are Banach spaces. If Y is separable it is enough to check (2.24) for a countable dense set of z, which gives a basis for algorithms to test whether $y \in \operatorname{im} T$. A different criterion based on inequalities involving duality was suggested in [13, Sect. 5].

3. EXAMPLES

3.1

We look first at the case where X is the Hardy space $H^2(\mathbb{D})$ discussed in the Introduction, and E is a Borel set $\subset \mathbb{D}$ such that the restriction map

$$T: \quad f \mapsto f|_E \tag{3.1}$$

is a bounded map from $H^2(\mathbb{D}) \to L^2(E; d\mu)$. For $\varphi \in L^2(E; d\mu)$ we can compute $T^*\varphi$ from

$$\langle T^*\varphi, k\rangle = \langle \varphi, Tk\rangle, \qquad k \in H^2(\mathbb{D}),$$

where for k we take

$$k_\zeta = k_\zeta(z) = (1 - \bar{\zeta}z)^{-1}, \qquad \zeta \in \mathbb{D}$$

the "reproducing element" at ζ (this procedure to calculate T^* is applicable *wherever X is a Hilbert function space with reproducing kernel* (r.k.)), and we get

$$(T^*\varphi)(\zeta) = (2\pi)^{-1} \int_E (1 - \bar{z}\zeta)^{-1}\, \varphi(z)\, d\mu(z).$$

Thus, T^* is an integral operator. If, for example, $E = (-1, 1)$ and μ is onedimensional Lebesgue measure, the abstract considerations of Section 2 revolve about the spectral resolution of the self-adjoint operator $B = TT^*$ on $L^2(-1, 1)$ defined by

$$(B\varphi)(x) = (2\pi)^{-1} \int_{-1}^{1} (1 - xx')\, \varphi(x')\, dx'.$$

This problem, as well as the analogous problem for $L^2(0, 1)$ have been completely solved by Mats Lindberg (unpublished), who also has partial results on the (compact) case where

$$Y = L^2(-a, a), \qquad 0 < a < 1.$$

If E is a set with interior, for example a subdomain of $\mathbb{D}$, then the natural "target space" Y would be a space of analytic functions on E, for example the Hardy space $H^2(E)$, or other Hilbert space (like a Bergman, or weighted Bergman) space, so chosen that (3.1) has dense range. (To handle some problems where $\bar{E}$ does not lie in $\mathbb{D}$ it would be desirable to extend the theory in Section 2 to unbounded T).

3.2

Denoting by $\mathbb{T}$ the unit circle, let $X = H^2(\mathbb{T})$, the Hardy space of functions in $L^2(\mathbb{T})$ whose Fourier coefficients of negative index vanish. Let $E \subset \mathbb{T}$ be a measurable set of positive Haar measure and $Y = L^2(E; d\theta)$ where $d\theta$ denotes Haar measure. Let

$$T: f \mapsto f|_E$$

be the restriction map. To calculate T^* it is convenient to introduce the operator P which projects $L^2(\mathbb{T})$ orthogonally on $H^2(\mathbb{T})$. For $\varphi \in L^2(E; d\theta)$ we get

$$\int_0^{2\pi} (T^*\varphi)(\theta)\, \overline{f(\theta)}\, d\theta = \int_E \varphi(\theta)\, \overline{f(\theta)}\, d\theta = \int_0^{2\pi} \tilde{\varphi}(\theta)\, \overline{f(\theta)}\, d\theta,$$

where $\tilde{\varphi} \in L^2(\mathbb{T})$ is gotten by extending φ to be zero on $\mathbb{T} \backslash E$. Hence $T^*\varphi - \tilde{\varphi}$ is orthogonal to $H^2(\mathbb{T})$, i.e.,

$$0 = P(T^*\varphi - \tilde{\varphi}) = T^*\varphi - P\tilde{\varphi}$$

whence

$$T^*\varphi = P\tilde{\varphi}. \tag{3.2}$$

For $f \in H^2(\mathbb{T})$ we get, putting $\varphi = Tf$ in (3.2),

$$T^*Tf = P(1_E f), \tag{3.3}$$

where 1_E is the characteristic function of E. Hence we see: *T^*T is the Toeplitz operator on $H^2(\mathbb{T})$ with symbol 1_E*. If we were to use a weight function in the target Hilbert space we would similarly get for T^*T Toeplitz operators with other kinds of symbols. Thus, restriction–extension problems in this set-up are equivalent to the spectral resolution of certain self-adjoint Toeplitz operators. Complete results are known, apparently, only when E is an interval (see, e.g., [6, 18]).

In an analogous way, when $X = L_a^2(D)$, the square-integrable functions analytic on the plane domain D, and $Y = L_a^2(D_0)$ with $D_0 \subset D$, with T being the restriction map $f \mapsto f|_{D_0}$, T^*T is a "Bergman–Toeplitz" operator (expressible as an integral operator since X has r.k.).

3.3

Perhaps the best studied example is that due to Slepian and Pollak [14], where $X = B_W$, the space of "band-limited functions," i.e., the set of

$$f(t) = \int_{-W}^{W} F(\omega)\, e^{it\omega}\, d\omega$$

for some $F \in L^2(-W, W)$. Here W is a positive parameter. The norm in B_W is given by

$$\|f\|^2 = (2\pi)^{-1} \int_{-\infty}^{\infty} \|f(t)\|^2\, dt = \int_{-W}^{W} |F(\omega)|^2\, d\omega.$$

If $\tau > 0$ is another positive parameter we take $Y = L^2(-\tau, \tau)$ and T is the restriction operator $f \mapsto f|_{(-\tau,\tau)}$.

Here B_W is a r.k. Hilbert space with reproducing element

$$K_s \equiv K_s(t) = (\pi^{-1})\, \frac{\sin W(t-S)}{t-s}$$

and calculation as in paragraph 3.1 gives, for $\varphi \in L^2(-\tau, \tau)$,

$$(T^*\varphi)(t) = \pi^{-1} \int_{-\tau}^{\tau} \frac{\sin W(t-s)}{t-s} \cdot \varphi(s)\, ds, \qquad t \in \mathbb{R}. \tag{3.4}$$

In the analysis of Pollak–Slepian the adjoint operator T^* is not introduced explicitly, instead they work in terms of a "time-limiting operator"

$$Z_\tau : f \mapsto 1_{(-\tau,\tau)} \cdot f \tag{3.5}$$

on $L^2(\mathbb{R})$ and "band-limiting operator" on $L^2(\mathbb{R})$ defined by

$$(L_W f)(t) = \int_{-W}^{W} \hat{f}(\omega)\, e^{it\omega}\, d\omega = \pi^{-1} \int_{-\infty}^{\infty} \frac{\sin W(t-s)}{t-s} \cdot f(s)\, ds, \tag{3.6}$$

where

$$\hat{f}(\omega) := (2\pi)^{-1} \int_{-\infty}^{\infty} f(t)\, e^{-it\omega}\, dt$$

(their notations are somewhat different).

From (3.4), (3.5), (3.6) we see that

$$T^* T = L_W Z_\tau$$

so that their analysis, based on spectral resolution of the self-adjoint operator $L_W Z_\tau$ on the Hilbert space B_W, which is an integral operator with kernel

$$(\pi^{-1})\, 1_{(-\tau,\tau)} \cdot \frac{\sin W(t-s)}{t-s},$$

is identical with that to which our general point of view would lead in this situation.

It is remarkable that in this case (where T^*T is compact) the eigenfunctions turn out to be the eigenfunctions of a certain Sturm–Liouville problem. The same thing happens in the problems referred to in subsection 3.1 studied by Lindberg, and several other analogous problems. This leads to the following general question: *Suppose* $Y = L^2(J; dz)$ *where* J *is some interval on* $\mathbb{R}$ *(*X *being arbitrary). For which* $T: X \to Y$ *does the second commutator of* TT^* *contain a nontrivial differential operator*? We recall that the "functions of TT^*," i.e., operators defined by

$$\int f(\lambda)\, dF(\lambda),$$

where f is a real-valued, and in general unbounded, Borel function comprise (with suitable assumptions, see [10, Sect. 127–129]) all densely defined closed self-adjoint operators on Y which commute with all operators commuting with TT^* (so-called second commutator of TT^*). It is remarkable that, in the special situations just enumerated, this functional calculus includes nontrivial second-order differential operators and it would be of interest to know just how typical, or exceptional, this is.

4. Further Horizons

Apart from the last-mentioned question there is another rather general question that can be raised in connection with Theorem 1. It is well known that, besides the "classical" form of the spectral theorem we have used, there is another formulation based on the point of view of "diagonalizing" a self-adjoint operator, i.e., setting up a unitary equivalence between it and an operator of multiplication by a real-valued function on a suitable L^2 space.

This point of view leads most naturally not to the family of projections $E(\lambda)$ as in the classical spectral theorem but rather to expansions of elements in the original Hilbert space in terms of "singular eigenfunctions" that do not belong to that space. This gives in principle the possibility, in "restriction" problems where T^*T is not compact, to formulate results analogous to those in Theorem 1 in terms of (generalized) eigenfunctions expansions, which are more tangible than the "resolutions of the identity" that we have employed. We should thus expect a theorem formally similar to the "abstract Bergman theorem" of [13], restoring the role of doubly orthogonal eigenfunctions but with infinite series replaced by integral transforms. The results achieved by Krein and Nudelman [6], van Winter [18], Lindberg and others in various special problems point strongly to the existence of this general theorem, to which I hope to return on another occasion. (A somewhat old, but useful introduction to generalized eigenfunctions is L. Gårding's article "Eigenfunction expansions" in [1].)

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